The Thermal Stresses in Spherical Shells Concentrically Heated.

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Introduction.

1. In recent years there have been a number of failures of materials when unequally heated which appeared to be due to the stresses set up by the differences of temperature in the materials. These stresses, although unimportant for small differences of temperature and at low temperatures, may be serious for large differences at high temperatures, owing to their increase with the temperature differences and to the great reduction of the elasticities and ultimate strengths of materials at high temperatures. It is therefore desirable, when possible, to calculate the thermal stresses to which structures are liable to be subjected by the differences of temperature likely to exist in them, especially if these differences are large and the temperatures themselves high.

The circumstances producing high thermal stresses are present to an exceptional degree in metallurgical, pottery and glass furnaces, and it is in these cases that disintegration of the materials used in the construction of the furnaces is most marked. The furnaces themselves vary so much in shape and material that it is advisable at present to discuss only some simple form and material, the investigation of which presents the minimum number of mathematical difficulties.

The roughly hemispherical form of many furnaces suggests the consideration of the comparatively simple case of a concentric spherical shell concentrically heated, composed of materials for which the physical constants over the ranges of temperature considered may be taken as approximately constant.

The case of a solid sphere concentrically heated was treated by Hopkinson* in 1879 and by Almansi† in 1897, on the assumption that, throughout the range of temperature considered, the expansion of the heated material was proportional to the rise of temperature.

It is, however, possible to cover those cases in which the differences of temperature are too great to allow this assumption to be made, by dealing in the equations connecting strain and stress directly with the expansion of the material from some standard temperature to the actual temperature,

* Hopkinson, J., 'Mess. of Maths.,' vol. 8, p. 168 (1879).

† Almansi, 'Atti Accad. Torino,' vol. 32, p. 701 (1897).

instead of introducing the temperature and the coefficient of expansion. In the following deduction this more general method is adopted.

Expansion of the Shell.

2. Let r (fig. 1) be the distance of a point from the centre of curvature of a concentric spherical shell at a uniform temperature and subjected to no stress, and let the temperature at all points of a thin shell of radius r be raised by θ, where θ is a function of r, and the pressure at the inside surface of the shell of radius r_1 become R_1, and that at the outside surface $r_2 = R_2$.

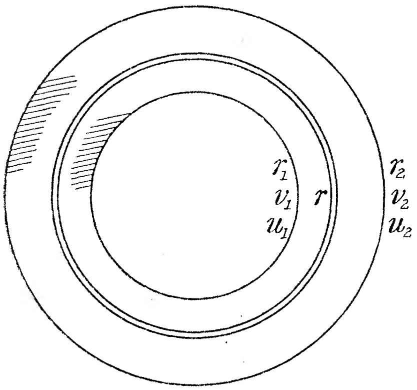

Fig. 1.—The Shell.

If the law of linear expansion of the material of the shell is expressed by the formula $l_\theta = l_0(1+a\theta+b\theta^2+c\theta^3$, etc.), where a, b, c are constants, we may write $l_\theta = l_0(1+a_\theta)$, where $a_\theta = a\theta+b\theta^2+c\theta^3+$ etc., and a_θ is then the actual expansion in length of a rod initially 1 cm. long when its temperature is raised by θ. In the same way, the change of volume may be represented by the formula $v_\theta = v_0(1+\alpha_\theta)$, where α_θ is the dilatation of 1 c.c. when its temperature is raised by θ° C.

If a thin concentric shell of internal radius r and external $r+dr$ be removed from the main shell while at its initial temperature, have its temperature raised so that the dilatation of 1 c.c. is α_θ, its volume, $4\pi r^2 dr$, will be increased by $4\pi r^2 \alpha_\theta dr$.

If each elementary shell of the material be isolated in this way and suffer dilatation at its own special rate, the whole increase in volume of the elementary shells will be $\int_{r_1}^{r_2} 4\pi r^2 \alpha_\theta \, dr$ and the mean dilatation of the material of the shell per c.c. $\dfrac{\int_{r_1}^{r_2} 4\pi r^2 \alpha_\theta \, dr}{\frac{4}{3}\pi(r_2^3-r_1^3)}$.

In general, the expanded shells would no longer fit together, and it is in forcing a fit that the stresses known as thermal stresses arise.

We note, in the first instance, that if the dilatation per cubic centimetre were the same for each shell, they would still fit together, and the thermal stresses would be zero. We therefore decompose the dilatation per cubic centimetre, α_θ of each elementary shell in the general case into two parts:

first, a part, $\bar{\alpha}$, equal to the mean dilatation for the whole shell; second, a part, α_θ', equal to the excess of the actual dilatation over that mean. The first part of the dilatation introduces no stresses; the thermal stresses are all due to the second part.*

THE STRESSES IN THE SHELL.

3. To find the nature and magnitude of the stresses and strains in the shell. Consider a small portion of the shell bounded by concentric spherical surfaces of radii r and $r+dr$ respectively, and by the surface of a cone of small vertical angle with its apex at the centre of curvature of the shell (fig. 2). Let R be

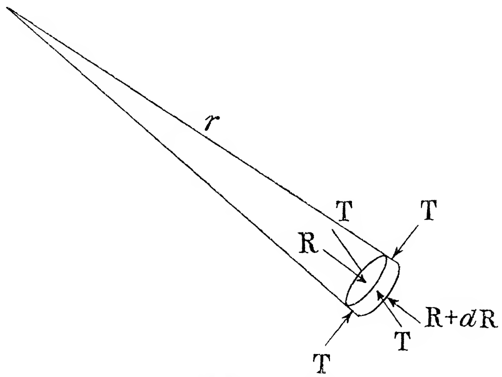

FIG. 2.—The pressures on an elementary disc.

the radial pressure on the element of the surface of the disc having radius r, and $R+dR$ that on the surface $r+dr$. Let T be the tangential pressure on the conical surface of the disc. These are by symmetry the only stresses. Let r' be the radial displacement outwards from the centre of curvature of any point initially at distance r from that centre. Again by symmetry this is the only displacement.

The stress-strain equations for the elementary disc are :—

$$(1-\sigma)T-\sigma R = -\epsilon\left(\frac{r'}{r}-a_\theta'\right), \tag{3.1}$$

$$R-2\sigma T = -\epsilon\left(\frac{dr'}{dr}-a_\theta'\right), \tag{3.2}$$

where ϵ is Young's Modulus, σ Poisson's ratio, supposed to be constants, and a_θ' the excess of the linear expansion of the material of the disc over the mean expansion for the whole of the shell.

* The absence of this limitation renders the equations of Rayleigh, 'Phil. Mag.,' vol. 1, p. 170, No. 3 *et seq.* (1901); and of Love, 'Theory of Elasticity,' 2nd ed., p. 106, No. 34, too general.

The equation for equilibrium of the forces acting on the disc is

$$2\mathrm{T}r = \frac{d}{dr}(\mathrm{R}r^2). \tag{3.3}$$

Substituting the value of T from (3.3) in the former equations they become

$$(1-\sigma)\frac{1}{2r}\cdot\frac{d}{dr}(\mathrm{R}r^2)-\sigma\mathrm{R} = -\epsilon\left(\frac{r'}{r}-\alpha_\theta'\right) \tag{3.4}$$

and

$$\mathrm{R}-\sigma\frac{1}{r}\frac{d}{dr}(\mathrm{R}r^2) = -\epsilon\left(\frac{dr'}{dr}-\alpha_\theta'\right). \tag{3.5}$$

Multiplyin_ (3.4) through by r, differentiating with respect to r and adding the result to (3.5) we eliminate r' and obtain on reduction

$$r^2\frac{d^2\mathrm{R}}{dr^2}+4r\frac{d\mathrm{R}}{dr} = \frac{2\epsilon}{1-\sigma}\cdot r\frac{d\alpha_\theta'}{dr} \tag{3.6}$$

the solution of which is

$$\mathrm{R} = \mathrm{A}+\frac{\mathrm{B}'}{r^3}+\frac{2\epsilon}{1-\sigma}\frac{1}{r^3}\int^r r^2\alpha_\theta'\,.\,dr, \tag{3.7}$$

where A and B′ are constants whose values depend on the pressures exerted on the inner and outer surfaces of the shell.

This expression for R suggests the substitution of v the volume of the shell of radius r instead of r in the equation, which becomes

$$\mathrm{R} = \mathrm{A}+\frac{\mathrm{B}}{v}+\frac{2}{9}\frac{\epsilon}{1-\sigma}\cdot\frac{1}{v}\int^v \alpha_\theta'\,.\,dv,$$

or

$$\mathrm{R}v = \mathrm{A}v+\mathrm{B}+\frac{2}{9}\frac{\epsilon}{1-\sigma}\int^v \alpha_\theta'\,dv. \tag{3.8}$$

If R_1 is the pressure on the inner surface r_1 of volume v_1, and R_2 that on the outer r_2 of volume v_2 we have

$$\mathrm{R}_1v_1 = \mathrm{A}v_1+\mathrm{B}+\frac{2}{9}\frac{\epsilon}{1-\sigma}\int^{v_1}\alpha_\theta'\,.\,dv$$

and

$$\mathrm{R}_2v_2 = \mathrm{A}v_2+\mathrm{B}+\frac{2}{9}\frac{\epsilon}{1-\sigma}\int^{v_2}\alpha_\theta'\,dv.$$

On eliminating A and B from these equations and reducing we have

$$\mathrm{R} = \mathrm{R}_2+(\mathrm{R}_1-\mathrm{R}_2)\frac{v_2/v-1}{v_2/v_1-1}-\frac{2}{9}\cdot\frac{\epsilon}{1-\sigma}\cdot\frac{1}{v}\int_v^{v_2}\alpha_\theta'\,dv,$$

writing E′ for the expansion $\int_{v_1}^{v}\alpha_\theta'\,dv = -\int_v^{v_2}\alpha_\theta' dv,$ due to the rise of temperature of that part of the shell within the spherical surface of volume v we have

$$\mathrm{R} = \mathrm{R}_2+(\mathrm{R}_1-\mathrm{R}_2)\frac{v_2/v-1}{v_2/v_1-1}+\frac{1}{9}\frac{\epsilon}{1-\sigma}\cdot 2\frac{\mathrm{E}'}{v}, \tag{3.9}$$

or if $u = 1/v$

$$R = \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} + (R_1 - R_2) \frac{u}{u_1 - u_2} + \frac{1}{9} \frac{\epsilon}{1-\sigma} \cdot 2uE'. \tag{3.9'}$$

To determine T we have from equation (3.3) $T = R + \frac{r}{2}\frac{dR}{dr}$, and on introducing the volume v instead of the radius r,

$$T = R + \frac{3}{2} v \frac{dR}{dv}. \tag{3.10}$$

On substituting the values of R and dR/dv in this equation and reducing we get

$$T = R_2 - (R_1 - R_2) \frac{v_2/2v + 1}{v_2/v_1 - 1} + \frac{1}{9} \frac{\epsilon}{1-\sigma} \left(3\alpha_\theta' - \frac{E'}{v} \right), \tag{3.11}$$

or in terms of u

$$T = \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} - \frac{R_1 - R_2}{2} \frac{u}{u_1 - u_2} + \frac{1}{9} \frac{\epsilon}{1-\sigma} (3\alpha_\theta' - uE'). \tag{3.11'}$$

The "mean pressure" or "equivalent hydrostatic pressure" to which the material of the shell at any point is subjected is $(R+2T)/3$ and the preceding equations give us

$$\frac{R+2T}{3} = \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} + \frac{1}{9} \frac{\epsilon}{1-\sigma} \cdot 2\alpha_\theta'. \tag{3.12}$$

The two shear stresses in diametral planes at right angles to each other are $(R-T)/3$ and our equations give

$$\frac{R-T}{3} = \frac{R_1 - R_2}{2} \cdot \frac{u}{u_1 - u_2} - \frac{1}{9} \frac{\epsilon}{1-\sigma} (\alpha_\theta' - uE'). \tag{3.13}$$

Writing v' for the increment of volume v of the spherical surface of radius r owing to the displacement r', and u' for the increment of $u = 1/v$, we have since $\frac{r'}{r} = \frac{1}{3}\frac{v'}{v} = -\frac{1}{3}\frac{u'}{u}$, on substituting the values of R and T in equation (3.1) and rearranging

$$\frac{r'}{r} = \frac{1}{3}\frac{v'}{v} = -\frac{1}{3}\frac{u'}{u} = -\frac{1-2\sigma}{\epsilon} \cdot \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2}$$
$$+ \frac{1+\sigma}{\epsilon} \cdot \frac{R_1 - R_2}{2} \frac{u}{u_1 - u_2} + \frac{1}{9} \frac{1+\sigma}{1-\sigma} \cdot \frac{E'}{v} \tag{3.14}$$

It may be noted that

$$\frac{1-2\sigma}{\epsilon} = \frac{1}{3k} \quad \text{and} \quad \frac{1+\sigma}{\epsilon} = \frac{1}{2\mu},$$

where k is the bulk modulus and μ the rigidity of the material.

The connection between r'/r and R is obtained readily by eliminating T and α_θ' from the original equations which gives $\frac{d}{dr}\left(\frac{r'}{r}\right) = \frac{1+\sigma}{2\epsilon}\frac{dR}{dr}$. (3.15)

The stresses and displacements due to the internal and external pressures R_1 and R_2 agree with Lamé's results.*

On putting $r_1 = 0$, $R_1 = R_2 = 0$ and $\alpha_\theta' = 3a\theta'$ in these equations, the thermal stresses reduce to Hopkinson's values for a solid sphere in which the range of temperature is small enough to allow the expansion with rise of temperature to be taken as uniform.

4. From these expressions it is seen that the cubical compression due to the pressures R_1 and R_2 on the surfaces of the shell is the same throughout the material,† while that due to the temperature distribution is at any point proportional to the excess of the thermal dilatation at that point over the mean thermal dilatation of the whole material of the shell.

The radial pressure R, the displacement r'/r, the tangential pressure T, and the shear stresses in diametral planes at right angles to each other, due to the pressures on the surfaces, are all linear functions of u the reciprocal of the volume v of the sphere through the point. The radial pressure R, and the displacement r'/r, due to the temperature distribution are proportional to the quotient of the excess cubical expansion of the elementary shells between the inner surface v_1 and the surface v, by the volume v of the latter surface. The other thermal stresses are found by the usual combinations of radial pressure and the mean pressure.

We see from the foregoing investigation that the general problem of the stresses in a spherical shell, produced by change of temperature such that isothermal surfaces are concentric spheres and by pressures applied to its surfaces, resolves itself into the determination by the methods indicated above of the stresses and displacements due to the expansion α_θ' of each elementary shell above or below the mean $\bar{\alpha}$ for the whole shell, and the superposition on these of the displacements due to the mean expansion $\bar{\alpha}$ and of the stresses and displacements due to the pressures R_1 and R_2 applied to the inner and outer surfaces of the expanded shell.

Graphical Representation of Stresses.

5. It is not easy to realise the meanings of the foregoing expressions for the stresses due to mechanical and to thermal causes without graphic representation and the expressions themselves suggest the following method of representing the stresses at a point, r, in terms of the volume, v, of the sphere through the point, or in the first case in terms of the reciprocal, u, of that volume.

* See *e.g.*, Love, 'Theory of Elasticity,' 2nd ed., pp. 139, 140 (1906).

† The constancy of the cubic compression due to the pressures on the two surfaces is noted by Lamb in his 'Statics,' p. 332 (1912).

A. *Stresses due to Pressures on Inside and Outside Surfaces.*

Set off along the axis of abscissæ (fig. 3) the reciprocals $OU_1 = u_1$ and $OU = u_2$ of the volumes v_1 and v_2 of inside and outside surfaces. At U_2 erect an ordinate, U_2R_2, equal on some convenient scale to the pressure, R_2, on the outside surface. Join the top, R_2, of this ordinate to the point U_1 and produce to cut the pressure axis in S_0. Through S_0 draw a line parallel to the axis of abscissæ, and let it cut the ordinates through U_1 and U_2 in S_1 and S_2.

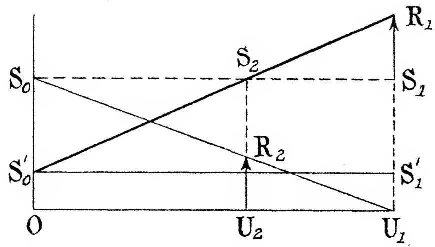

FIG. 3.—The mean pressure OS_0' determined by U_2R_2, $U_1R_2S_0$, S_0S_1, S_1R_2, and $R_1S_2S_0'$.

Set off above S_1 a length, S_1R_1, equal to R_1, the pressure on the inside surface, and join the end, R_1, of this ordinate to S_2, and produce to cut the axis of pressures in S_0'.

Then OS_0 is $R_2 \dfrac{u_1}{u_1-u_2}$, S_0S_0' is $R_1 \dfrac{u_2}{u_1-u_2}$: hence

$$OS_0' = \frac{R_2u_1 - R_2u_1}{u_1-u^2}, \tag{5.1}$$

that is the "mean pressure" or "mean hydrostatic pressure" throughout the shell.

To get the terms involving $(R_1-R_2)\dfrac{u}{u_1-u_2}$ necessary to express the various stresses, erect at U_1, fig. 4, the ordinate $U_1A = (R_1-R_2)$, join the

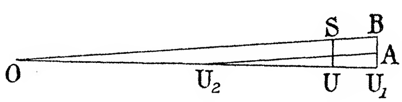

FIG. 4.—$U_1A = R_1 - R_2$, OB is parallel to U_2A_1, US gives the stress at U.

extremity A of it to the point U_2 and draw through O a straight line OB, parallel to this join, cutting U_1A in B. The ordinate US at any point U is then $(R_1-R_2) \cdot \dfrac{u}{u_1-u_2}$. Hence the sheer stress in each of two mutually perpendicular diametral planes through the point U

$$= \tfrac{1}{2} \cdot US. \tag{5.2}$$

For the other stresses we have

$$R' = \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} + (R_1 - R_2)\frac{u}{u_1 - u_2} = OS_0' + US, \tag{5.3}$$

$$T = \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} - \frac{R_1 - R_2}{2}\,\frac{u}{u_1 - u_2} = OS_0' - \tfrac{1}{2}US, \tag{5.4}$$

and for the displacements

$$\left.\begin{aligned} \frac{r'}{r} = \frac{1}{3}\frac{v'}{v} &= -\frac{1-2\sigma}{\epsilon}\,\frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} + \frac{1+\sigma}{\epsilon}\,\frac{R_1 - R_2}{2}\,\frac{u}{u_1 - u_2} \\ &= -\frac{1-2\sigma}{\epsilon}OS_0' + \frac{1+\sigma}{\epsilon}\,\tfrac{1}{2}US \end{aligned}\right\}. \tag{5.5}$$

B. *Stresses due to the Deviation of the Dilatation due to Temperature from the mean Dilatation.*

6. Set off (fig. 5) as abscissæ OV_1, OV_2, the volumes, v_1 and v_2 of the inner and outer surfaces of the shell. Take as ordinates the values of E′, the increase of volume of the series of infinitely thin shells between v_1 and v due to the excess of the temperature of each shell over the mean temperature of the whole of the shells between the inner and outer surfaces.

We thus get the curve of volume increase, V_1, PV_2, extending from V_1, where it cuts the axis of volumes, to V_2, where it again cuts the axis. As we proceed outwards it slopes upwards, where the layers are hotter, or the dilatations greater, than the mean, and downwards, where they are colder, or where the dilatation is below the mean. The slope upwards at any point is proportional to the excess dilatation per cubic centimetre on the scale taken.

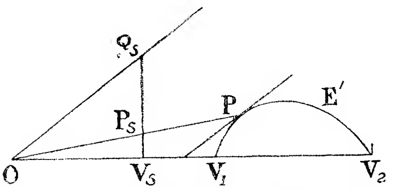

Fig. 5.—E′ the excess expansion curve gives at P with OQ_s parallel to the tangent at P, the lines V_sP_s, V_sQ_s determining the stresses.

Join P, a point on the curve, to the origin, O. Through O draw OQ_s parallel to the tangent to the E′ curve at P, and at V_s, corresponding to a volume v_s, a convenient multiple of 10, erect a perpendicular to the volume axis, cutting OQ_s in Q_s and OP in P_s.

Then the radial pressure R at the point P is

$$\frac{1}{9}\,\frac{\epsilon}{1-\sigma}\cdot 2\,\frac{V_sP_s}{v_s}, \tag{6.1}$$

the " mean pressure " at P is

$$\frac{1}{9}\,\frac{\epsilon}{1-\sigma}\cdot 2\,\frac{V_sQ_s}{v_s}, \tag{6.2}$$

the tangential pressure T is

$$\frac{1}{9}\frac{\epsilon}{1-\sigma}\frac{3V_sQ_s-V_sP_s}{v_s}, \tag{6.3}$$

the two shear stresses are

$$-\frac{1}{9}\frac{\epsilon}{1-\sigma}\cdot\frac{P_sQ_s}{v_s}, \tag{6.4}$$

and the radial displacement is given by

$$\frac{r'}{r}=\frac{1}{3}\frac{v'}{v}=\frac{1}{9}\frac{1+\sigma}{1-\sigma}\cdot\frac{V_sP_s}{v_s}. \tag{6.5}$$

To the latter must be added in case the mean temperature has been raised $\bar{\theta}$ the radial displacement given by $\frac{r'}{r}=\bar{a}=\frac{1}{3}\bar{\alpha}$ due to it.

Construction of the Excess Expansion Curve E′.

7. The problem of determining the thermal stresses at a point in the material of the shell thus reduces to the construction of the curve E′, and the drawing of the radius vector and tangent to the curve at the point.

To draw the curve most conveniently we have $E'=\int_{v_1}^{v}\alpha_\theta'\,dv$, where α_θ' is the excess of the cubical dilatation of 1 c.c. at θ, over the mean dilatation $\bar{\alpha}$ for the whole shell.

Hence if α_θ is the actual dilatation at θ,

$$E'=\int_{v_1}^{v}(\alpha-\bar{\alpha})\,dv=\int_{v_1}^{v}\alpha_\theta\,dv-\bar{\alpha}\,(v-v_1),$$

or if $\quad E=\int_{v_1}^{v}\alpha_\theta\,.\,dv \quad$ and $\quad E_2=\int_{v^1}^{v_2}\alpha_\theta\,.\,dv=\bar{\alpha}\,(v_2-v_1),$

$$E'=E-E_2\frac{v-v_1}{v_2+v_1}. \tag{7.1}$$

We, therefore, proceed as follows:—Draw first the curve of temperature distribution, θ (fig. 6), throughout the shell as determined by the temperatures at the two surfaces and the laws of conductivity.

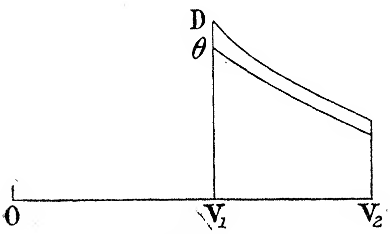

Fig. 6.

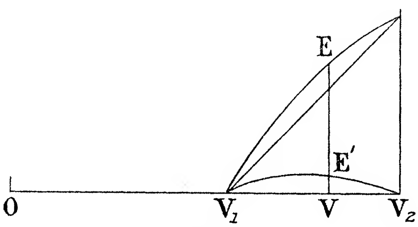

Fig. 7.

Fig. 6.—θ the curve of temperature, D the curve of dilatation.
Fig. 7.—The excess expansion curve E′ obtained from the expansion curve E.

For each temperature estimate from tables of the expansion of the material of the shell the dilatation, α_θ, of 1 c.c. at each part of the shell, and draw the curve of dilatation per cubic centimetre, D, to correspond to the temperature curve. Find the total expansion, E, that is the area included between the axis of volume, the D curve, the ordinate through v_1 and that through v for different values of v up to v_2, and plot the values in terms of v, curve E (fig. 7).

Join the end of the curve at v_2 with its starting point at v_1. The excess of the ordinate of the curve at any point over the ordinate of the join is the ordinate of the excess cubical expansion curve, E′, required.

Deductions from the Excess Expansion Curve.

8. The curves of excess cubical expansion, E′ in terms of v allow the conditions for zero and maximum values of the various stresses to be seen by inspection in many cases. If the dilatation due to temperature decreases as we proceed outwards through the shell, the curve E′ will have one maximum between its zeros at v_1 and v_2, while if the temperature and dilatation increase outwards E′ will have one minimum between its zeros. If the dilatation is a maximum at some layer within the shell the curve, E′, on starting from its zero at v_1, is first negative, then passes through zero and becomes positive and ends at zero at v_2. If the dilatation is a minimum at a layer within the shell, the curve starts by being positive near its v_1 zero, passes through zero and becomes negative before reaching its v_2 zero.

In all cases the maximum and minimum ordinates of the E′ curve occur at the layers for which the dilatation due to temperature has the mean value for the whole shell. The ordinate is a maximum if the dilatation decreases outwards and a minimum if it increases.

The displacement is zero for points at which the E′ curve crosses the axis of volumes. Two of these points are at the inner and outer surfaces respectively. The maximum outwards displacement occurs at the layer for which the tangent to the E′ curve where it is positive passes through the origin, and the maximum inwards displacement at the layer for which the tangent to the curve where it is negative passes through the origin. Both layers are nearer to the centre of the sphere than the layers of maximum or minimum E′.

The radial pressure is also zero at points at which the E′ curve crosses the axis. Two of the points are at the inner and outer surfaces respectively. The maximum value of the radial pressure occurs in the same layer as the maximum outward displacement, and the maximum radial tension in the layer of maximum inward displacement. These layers are a little nearer the

centre of the sphere than those of maximum and minimum ordinates of the E′ curve respectively.

The mean pressure about a point vanishes at the layers for which the dilatation is the mean for the whole shell, that is at the maximum E′. The maximum values of the mean pressure occur in the layers for which the temperature dilatation differs most from the mean, that is, where the slope of the E′ curve is greatest. In most practical cases this will be at or near the inner and outer surfaces and at the layer within at maximum or minimum temperature if such exists. The mean pressure is positive in the hotter and negative (*i.e.*, a tension) in the cooler portions of the shell.

The two shears in diametral planes at right angles to each other are zero for the layers for which the tangent to the E′ curve passes through the origin, that is for the layers for which the displacements and the radial pressures are numerically greatest. They are greatest when the temperature dilatation is above the mean and is increasing outwards, or when it is below the mean and is decreasing outwards.

The tangential pressure is zero in a layer between that of mean dilatation and that of maximum radial pressure, for which the slope of the E′ curve is one-third that of the join of the point on the curve with the origin. In general the tangential pressure is positive and large at or near the layer at highest temperature, and negative (*i.e.*, a tension) and large at or near that at lowest temperature.

From the foregoing it is seen that the determination of the stresses produced in a spherical shell by differences of temperature concentric with the shell is reduced to that of calculating the expansions of the elementary layers of the shell due to the deviation of their temperature from the mean temperature of the whole shell.

THE TEMPERATURE DISTRIBUTION.

9. The determination of the distribution of temperature throughout the shell, given the temperatures of the two surfaces, is a problem in heat or temperature conductivity, which if the conductivity of the material were independent of the temperature, could be solved by the usual methods. If the thermal conductivity, k, of the material cannot be taken as independent of the temperature the condition for continuity of flow of heat in the material must be given the general form

$$c\rho \frac{\partial \theta}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(kr^2 \frac{\partial \theta}{\partial r} \right), \tag{9.1}$$

where c is the specific heat and ρ the density of the material.

The difficulty of determining θ as a function of the time and radius by

means of this equation, when c and k both vary considerably with temperature, is sufficient justification for making some simplifying assumptions in order to obtain some idea of the magnitudes of the thermal stresses in materials under common practical conditions.

The Steady State.

10. In the first instance let us assume that the temperature at each point of the material has become steady. Our equation reduces to $\frac{d}{dr}\left(kr^2\frac{d\theta}{dr}\right) = 0$, which on integration gives

$$k\frac{d\theta}{dr} = -\frac{H}{4\pi r^2}, \tag{10·1}$$

where H is the heat which flows outward through the shell per second.

The heat conductivity k in general decreases with increase of temperature,* but there appear to be some furnace materials for which it is approximately independent of temperature.†

Assuming that the shell is of such a material, we have for the temperature

$$\theta = A + \frac{H}{4\pi kr}.$$

If the volume $v_2 - v_1$ of the material of the shell is small compared to its internal volume v_1 this may be given the approximate form

$$\frac{\theta - \theta_2}{\theta_1 - \theta_2} = \frac{v_2 - v}{v_2 - v_1}, \tag{10.2}$$

where as before, θ_1 and θ_2 are the temperatures at the surfaces and the mean temperature $\bar{\theta} = \frac{\theta_1 + \theta_2}{2}$.

Taking the expansion of the material of the shell to be proportional to the excess of temperature over the mean, this gives for the expansion, E′, of the elementary shells between the inner surface, v_1, and that of volume v,

$$E' = \frac{\alpha}{2} \cdot \frac{\theta_1 - \theta_2}{v_2 - v_1} \cdot \left\{\left(\frac{v_2 - v_1}{2}\right)^2 - \left(\frac{v_2 + v_1}{2} - v\right)^2\right\}. \tag{10.3}$$

The maximum value of E′ is $\frac{\alpha}{2}(\theta_1 - \theta_2)\frac{v_2 - v_1}{4}$ and occurs at $v = \frac{v_2 + v_1}{2}$.

The maximum R is approximately

$$\frac{1}{9}\frac{\epsilon}{1-\sigma} \cdot \frac{\alpha}{2}(\theta_1 - \theta_2)\frac{v_2 - v_1}{v_2 + v_1}, \tag{10.4}$$

* Lees, 'Phil. Trans. Roy. Soc.,' A, vol. 191, p. 399 (1898); Eucken, 'Ann. der Phys.,' vol. 34, pp. 217, 219 (1911).

† Clement and Egy, 'Univ. of Illinois Bulletin,' vol. 6, No. 42 (1909).

and occurs slightly within the middle layer of the shell. It is a pressure if $\theta_1 > \theta_2$ and a tension if $\theta_1 < \theta_2$.

The maximum T is approximately

$$\frac{1}{9}\frac{\epsilon}{1-\sigma}\cdot\frac{3\alpha}{2}(\theta_1-\theta_2), \tag{10.5}$$

and is found at the surfaces. At the inside it is a pressure, at the outside surface a tension if $\theta_1 > \theta_2$.

The maximum value of the mean pressure about a point is

$$\frac{1}{9}\frac{\epsilon}{1-\sigma}\cdot\alpha(\theta_1-\theta_2), \tag{10.6}$$

and occurs at the surfaces.

The maximum shears are

$$\frac{1}{9}\frac{\epsilon}{1-\sigma}\cdot\frac{\alpha}{2}(\theta_1-\theta_2) \tag{10.7}$$

at the surfaces.

The maximum displacement is

$$\frac{1}{9}\frac{1+\sigma}{1-\sigma}\cdot\frac{\alpha}{4}(\theta_1-\theta_2)\frac{v_2-v_1}{v_2+v_1}, \tag{10.8}$$

and occurs approximately at the middle layer of the shell.

Numerical Values : Practical Steady Case.

11. When we attempt to assign numerical values to these quantities in any practical case, we are met by the difficulty that we have little or no knowledge of the expansion of the materials of furnaces at high temperature, and no knowledge whatever of their elastic constants and ultimate strengths at high temperatures.

If the wall of the furnace is 20 cm. thick, and the temperature inside is 1100° C., and outside 100° C. $\theta_1-\theta_2$ is 1000° C., and, if we take the wall to be of material of about the physical properties of firebrick at ordinary temperatures, we should have the coefficient of cubical expansion 12×10^{-6}, Young's modulus about 3×10^{11}, Poisson's ratio about 1/3, a crushing strength of about 5×10^{8}, and a tensile strength of about 2×10^{8} dynes per square centimetre.* At higher temperatures the expansion is greater, Young's modulus less, the crushing strength less, and the tensile strength probably considerably less than at ordinary temperatures.

Calculating the stresses on the basis of the above figures we have for the maximum :—

Radial pressure $R = 3\times10^{8}\cdot\frac{v_2-v_1}{v_2+v_1}$ near the middle of the wall.

* From Mark's 'Mechanical Engineer's Handbook.'

Tangential pressure $T = 3 \times 10^8$, positive at the inside surface and negative, that is a tension, at the outside surface.

Mean pressure at a point $= 2 \times 10^8$, positive at the inside, negative at the outside surface.

Shear stress $= 1{\cdot}0 \times 10^8$ at the surfaces.

Displacement $= 0{\cdot}7 \times 10^{-3} \frac{v_2 - v_1}{v_2 + v_1}$ near the middle of the wall.

These maxima exceed the limits of tensile strength of the material, and cracks would be formed on the outer surface, even if the wall were solid. If the bricks were set in cement, the cracks would form at the joints.

Unsteady States.

12. If the temperature throughout the shell is not steady, the E′ curves are much more complicated in shape, and some of the stresses may be considerably greater than they are in the steady state.

Suppose, for example, that the shell is all at 100° C. and the inner surface is suddenly raised to 1100° C. After the first few seconds, the E′ curve, corresponding to the temperature distribution, starts from the v_1 zero at an angle to the axis of volume whose tangent is double that for the steady state. The maximum ordinate of the curve is, however, much smaller than in the steady state, and the inclination of the curve to the axis of volume at the v_2 zero is also much smaller. Hence the tangential pressure, T, the shear stresses, and the mean pressure at the inner surface, are approximately double those in the steady state at the inner surface, and the tangential tension at the outer surface smaller than that in the steady state. The radial pressure, R, and the radial displacement do not exceed the values attained in the steady state.

As a second case, suppose the shell, after attaining the steady state with inside temperature 1100° C. and outside 100° C., has its inside surface suddenly cooled to 100° C. The E′ curve at a second or two after the change proceeds from the inner V_1 zero, with a downward slope equal to the upward slope in the steady state. The tangential pressure at the inner surface in the steady state becomes an equal tangential tension, the mean pressure and the shear stresses are reversed, while the maximum radial pressure and the displacement retain approximately the values they had in the steady state. At the outer surface the stresses change from their steady values slowly.

Thus a sudden rise of the temperature of the inside of the shell from its initial value, supposed to be that of the outside surface, introduces at the inner surface double the tangential pressure, the mean pressure and the

shear stresses which exist at that surface in the steady state, hence spalling may result.

A sudden fall of the inside temperature from that of the steady state to that of the outside surface converts the tangential pressure of the steady state at that surface into an equal tension, the mean pressure into an equal tension, and reverses the sign of the shear stress.

Although the tangential pressure at the ner surface in the steady state may not have been sufficient to cause crushing, the tension thus produced may cause radial cracks, owing to the tensile strength of the material being less than its crushing strength.

Furnaces.

13. Through the kindness of Sir Robert Hadfield and of Dr. J. W. Mellor I have been able to inspect furnaces which have been in use for some time and to examine fractured materials, and the appearances of the cracks in the bricks or other materials of the furnaces show that the above theory reproduces the principal facts. The general tendency of rapid internal heating to produce spalling of the inner surface, and of rapid cooling to produce radial cracks, is quite well known to those in charge of the furnaces, and proper steps are taken to reduce the rate of change of temperature when the furnace has to be heated or cooled. A furnace which is to be taken down or repaired is cooled rapidly by blowing cold air through it in order to produce disintegration of the walls and facilitate the taking-down process.

Conclusion.

14. It is thus evident that the stresses in the wall of a furnace of spherical or approximately spherical form produced by a distribution of temperature throughout the wall, which is either known from observation or calculable by the laws of conduction of heat, can be determined readily by graphical or other methods if the law of expansion of the material throughout the range of temperature is known, and Young's modulus and Poisson's ratio or any two elastic moduli are also known, and can be taken as approximately constant. Whether the stresses thus determined will produce crushing or rupture of the wall depends on the values of the crushing load or the tensile strength of the material at the temperature which exists in the region of the stress. But the forms of the expressions for the stresses show that these stresses are proportional to the product of the dilatation of the material with rise of temperature into Young's modulus for the material. In seeking better refractory materials for furnace walls, we require to keep this product as low as possible. A low Young's modulus generally

means a correspondingly low tensile strength, but if a material could be found with a lower modulus, and a strength not reduced, it would, other things being equal, be superior. The search for a material with a low dilatation with rise of temperature seems more promising, and it is probably on account of its low dilatation that silica brick has proved so useful in furnace construction.

The Two-dimensional Slow Motion of Viscous Fluids.

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The present paper is a contribution to the treatment of problems which require a solution of the differential equation $\nabla^4\psi = 0$. Amongst such problems are to be found not only the very slow motions of a viscous fluid in two dimensions, but also the flexure of thin flat plates.*

The prosecution of the investigation has been made possible by the support of the Department of Scientific and Industrial Research, which has provided financial assistance to enable two of us to devote the whole of our time to the research, and our thanks are offered to the Department for its assistance. We also desire to acknowledge the facilities afforded by the Governing Body of the Imperial College of Science and Technology in placing a room at our disposal in the Department of Aeronautics.

The method of attack was suggested by the results of an earlier paper† on the solution of Laplace's equation, $\nabla^2\psi = 0$. It was there found that, for any forms of single or double boundary, the solution of problems on the torsion of cylinders or the irrotational motion of an inviscid fluid could be made to depend on a number of definite integrals. In general, the evaluation of these integrals involves the use of graphical and mechanical methods, and much progress has been made in the direction of simplification of the processes. It is hoped that a paper may be presented elsewhere in the near future showing the application to a number of engineering problems.

The solution of the equation $\nabla^4\psi = 0$ given in the present paper follows the earlier work in its generality as to boundary forms and also in the general dependence on graphical and mechanical integration for arithmetical

* Lamb, 'Hydrodynamics,' p. 604.

† 'Roy. Soc. Proc.,' A, vol. 95, pp. 457–475 (1919).